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Squeezed "atomic" states, pseudo-Hermitian operators and Wigner D-matrices

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Abstract. The states of N two-level atoms can be mapped onto the eigenvectors of angular momentum (with $j = N/2$) and this system in interaction with a radiation field constitutes a fundamental model in Quantum Optics. There from one may construct "atomic" coherent states and "minimum uncertainty packets". The "squeezing" of such states is of considerable contemporary interest. We show that the properties of squeezed atomic states are most elegantly and economically expressed in terms of "pseudo-Hermitian" operators and through Wigner D-matrices and their analytical continuation.

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1 Introduction

The Hamiltonian of an oscillator (or a mode of a field) is given in suitable units by $H = p^2/2+x^2/2$ with $[x, p] = i\hbar$. Defining $a = (x + ip)/\sqrt{2\hbar}$ and $a^\dagger = (x - ip)/\sqrt{2\hbar}$ one has $[a, a^\dagger] = 1$ and $H = (a^\dagger a + 1/2)\hbar$ and the underlying Hilbert space is mapped by vectors $|n\rangle, (n = 0, 1, 2, \dots)$ such that $a^\dagger a|n\rangle = n|n\rangle$ while $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n + 1\rangle$. With α a complex number the "coherent" state for this system given by $|\alpha\rangle = \exp(\alpha a^\dagger - \alpha^* a)|0\rangle = \exp(-|\alpha|^2/2)\sum_n(\alpha^n/\sqrt{n!})|n\rangle$ which is an eigenvector of a belonging to the eigenvalue α and enjoys the property that it is a minimum uncertainty packet viz. $\Delta x\Delta p = \hbar/2$ as $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\hbar/2}$ and $\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\hbar/2}$ for such a state. Another useful concept is introduced through a dilation of x viz. $x \to x' = e^\xi x$ where ξ is a real number,and concomitantly $p \to p' = e^{-\xi}p$ keeping intact the canonical commutation relation $[x', p'] = i\hbar$. This results in a squeezing (or reduction) in the spread or uncertainty in one of the variables at the cost, of course, in a corresponding increase in that of the other, retaining the minimal value of the product $\Delta x\Delta p$. A little algebra reveals that at the level of the annihilation (and creation) operators this dilation entails a canonical Bogoluibov transformation viz. $a \to b$ where $b = (\cosh\xi)a + (\sinh\xi)a^\dagger$ which, furthermore, is easily seen to be implemented via a unitary transformation viz. $b = \exp[(\xi/2)(a^2 - a^{\dagger 2})]a\exp[-(\xi/2)(a^2 - a^{\dagger 2})]$. The notions introduced above have played an important role in Quantum Optics.

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The physics of a two-level atom is most conveniently expressed in terms of the mathematics of the two states of a spin one-half particle, and correspondingly a system of N such atoms is describable as the symmetric states of N spin one-half particles viz. states pertaining to angular momentum $\mathbf{J}$ with $j = N/2$ with the "magnetic" projection quantum number m given by one-half the difference in the number of atoms in the excited and ground states. The interaction between radiation and a system of such atoms attracts considerable attention being a fundamental model in Quantum Optics. In particular much work has been done to extend the notions of coherent states and squeezing in the context of such systems of atoms (or equivalently to the states of angular momentum). From the commutation relations $[J_x, J_y] = iJ_z$ for the components of the angular momentum operator one readily derives the corresponding uncertainty relation $\Delta J_x\Delta J_y \geq |\langle J_z\rangle|/2$. For the state $|j, m\rangle$ the quantities ΔJ_x and ΔJ_y are easily found to be both equal to $\sqrt{[j(j + 1) - m^2]/2}$ and the uncertainty inequality translates into $[j(j + 1) - m^2] \geq |m|$. The inequality becomes an equality when $m = +j$ or $m = -j$, and these clearly would be the states of minimum uncertainty (for which $\Delta J_x = \Delta J_y = \sqrt{j/2}$ and $\Delta J_x\Delta J_y = j/2$). It may be noted[1] that for these special

[1] With $[\hat{A}, \hat{B}] = i\hat{C}$ the minimum uncertainty state $|\psi\rangle$ satisfying $\Delta A\Delta B = |\langle\hat{C}\rangle|/2$ are states for which $(\hat{A} + i\lambda\hat{B})|\psi\rangle = (\langle\hat{A}\rangle + i\lambda\langle\hat{B}\rangle)|\psi\rangle$ for some real λ. This is the condition under which the Schwartz inequality (used to prove the uncertainty relation from the commutator) reduces to an equality. Here $\hat{A} = J_x$ and $\hat{B} = J_y$ and $\lambda = \pm 1$, while for the oscillator we had $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}$ and $\lambda = -1$.

states $J_\pm|j, m = \pm j\rangle = 0$ where $J_\pm \equiv J_x \pm iJ_y$ which for this system are analogous to a and $a^\dagger$ for oscillators. However, there should be nothing particular about a given z-direction. Accordingly a rotation should yield states basically equivalent to these minimum uncertainty vectors. Such "atomic coherent states" were proposed by Bloch [1] and subsequently by Radcliffe and others [2,3] and can be defined as

$$|j,\zeta\rangle = Ne^{\zeta J_-}|j, m = +j\rangle = e^{\zeta J_- - \zeta^* J_+}|j, m = +j\rangle$$
$$= \frac{1}{(1+|\zeta|^2)^j}\sum_{n=0}^{2j}\sqrt{{}^{2j}C_n}\zeta^n|j, m = j-n\rangle \quad \text{(1a)}$$

where N is a normalization constant and ζ a complex number. Putting

$$\zeta = \tan(\theta/2)e^{i\phi} \quad \text{(1b)}$$

we have

$$\langle j,\zeta|J_z|j,\zeta\rangle = j\cos\theta, \qquad \langle j,\zeta|J_x|j,\zeta\rangle = j\sin\theta\cos\phi,$$
$$\langle j,\zeta|J_y|j,\zeta\rangle = j\sin\theta\sin\phi \quad \text{(1c)}$$

and thus we see that θ and ϕ are the polar and azimuthal angles which the average angular momentum vector for such a state ($\langle\mathbf{J}\rangle = \langle j,\zeta|\mathbf{J}|j,\zeta\rangle$) makes in the coordinate system. For such a state one can easily verify that

$$\Delta J_x = \sqrt{\frac{j}{2}}\sqrt{1-\sin^2\theta\cos^2\phi}, \quad \text{(2a)}$$

$$\Delta J_y = \sqrt{\frac{j}{2}}\sqrt{1-\sin^2\theta\sin^2\phi}, \quad \text{(2b)}$$

$$\Delta J_z = \sqrt{\frac{j}{2}}\cos\theta. \quad \text{(2c)}$$

One must not, however, make the mistake that as $\Delta J_x < \sqrt{j/2}$ some squeezing has been introduced. The "atomic" coherent state $|j,\zeta\rangle$ is nothing but a rotated minimum uncertainty state and clearly a mere change of axes cannot introduce the quantum correlations involved in the act of squeezing. Suppose we rotate the axes in such a manner that the new z-axis (z' say) is in the direction of the average angular momentum vector such that

$$\langle J_{z'}\rangle = |\langle\mathbf{J}\rangle| = \sqrt{\langle J_x\rangle^2 + \langle J_y\rangle^2 + \langle J_z\rangle^2} \quad \text{(3a)}$$

and, therefore, x'-, y'-plane lies in the plane orthogonal to $\langle\mathbf{J}\rangle$. Accordingly we have

$$\langle J_{x'}\rangle = 0 = \langle J_{y'}\rangle, \quad \text{(3b)}$$

and thus with $[J_{x'}, J_{y'}] = iJ_{z'}$ one must have $\Delta J_{x'}\Delta J_{y'} \geq |\langle\mathbf{J}\rangle|/2$. The equality sign holds if the given state is a minimum uncertainty state. If in **such a frame** we have

$$\Delta J_{x'} \ \text{ or } \ \Delta J_{y'} < \sqrt{|\langle\mathbf{J}\rangle|/2} \quad \text{(3c)}$$

then we shall say that the atomic system is spin-squeezed.

A candidate for a squeezed atomic state which has been considered by several authors and which is in principle realisable through some nonlinear interaction between a system of effective two-level atoms and a suitable radiation field is obtainable as eigenstates of the operator

$$\Lambda = \left(e^{\xi}J_+ + e^{-\xi}J_-\right)/2 \quad \text{(4)}$$

with ξ real. These states can describe a system of N two-level atoms interacting with a squeezed radiation field [4] (the so-called squeezed vacuum) with a parameter ξ called the squeeze parameter[2]. We show that the handling of the underlying mathematics is most suitably and efficiently carried out through the introduction of pseudo-Hermitian operators and employing the analytical continuation of the Wigner D-matrices to imaginary angles.

2 Pseudo-Hermitian operators and squeezed atomic states

The notion of pseudo-Hermitian operators has been known since the early 1940's through the works [5] of Dirac, Pauli, Suraj Narayan Gupta, Bleuler, Sudarshan, Lee, and Wick. The interest in such matters has more recently been revived through a series of papers by Bender and his collaborators [6] on Hamiltonians which are invariant under the combined operation (PT) of parity and time reversal but separately violate P and T, but yet have real spectra. It was subsequently shown by Mostafazadeh [7] that the structure responsible for this feature is pseudo-Hermicity viz. the existence of a linear invertible Hermitian operator η which is such that while $H^\dagger \neq H$ but yet H enjoys the property that $H^\dagger = \eta H\eta^{-1}$. Under such circumstances either the spectrum of H is real or eigenvalues come in complex conjugate pairs. Such an operator H is said to be η-pseudo-Hermitian. The operator η entering the discussion is sometimes called a metric operator. This is because it may be used to define an inner product namely $\langle\psi,\phi\rangle_\eta \equiv \langle\psi,\eta\phi\rangle$ which is a genuine positive definite inner product if and only if η is a positive definite operator. It is also instructive to record that a pseudo-Hermitian operator has a complete set of bi-orthonormal eigenvectors $\{|\psi_n\rangle, |\phi_n\rangle\}$ with $H|\psi_n\rangle = E_n|\psi_n\rangle$ and $H^\dagger|\phi_n\rangle = E_n^*|\phi_n\rangle$ and $\langle\phi_m|\psi_n\rangle = \delta_{m,n}$ and $\sum_n|\psi_n\rangle\langle\phi_n| = I$. Furthermore, the necessary and sufficient condition for a pseudo-Hermitian operator with a discrete complete set of bi-orthonormal eigenstates to have only real eigenvalues is that it should be possible to express the operator η as $OO^\dagger$ where O is an invertible linear operator.

[2] It is the same parameter involved in the Bogoluibov transformation of the radiation field operators $a, a^\dagger$ described in the first paragraph of this section. ξ represents the degree of squeezing of the radiation field and is, therefore, called the squeeze parameter. Further, the process of squeezing the vacuum produces a number known as the average photon number of the squeezed vacuum $\bar{n}$ which is given by relation $\bar{n} = \sinh^2\xi$. The range of ξ is given by $0 \leq \xi < \infty$. Obviously, $\xi = 0$ represents the vacuum of the ordinary (un-squeezed) radiation field.

Let us now, in this framework, study the "squeezing" operator given by equation (4). Noting the effect of a rotation by an angle θ about the z-axis of the angular momentum operator J_x we have

$$e^{-iJ_z\theta}J_x e^{+iJ_z\theta} = \cos\theta J_x + \sin\theta J_y = \left(e^{-i\theta}J_+ + e^{+i\theta}J_-\right)/2 \quad (5)$$

wherein putting $\theta = i\xi$ we observe that

$$\Lambda = e^{\xi J_z}J_x e^{-\xi J_z}$$

or, in other words, the operator Λ is nothing but a "hyperbolically" rotated J_x viz rotation through an imaginary angle $i\xi$ about the z-axis. Thus Λ is a similarity (though not a unitary) transform of the Hermitian operator J_x and $\Lambda^\dagger \neq \Lambda$. However,

$$\Lambda^\dagger = \left(e^{\xi J_z}J_x e^{-\xi J_z}\right)^\dagger = e^{-\xi J_z}J_x e^{\xi J_z} = e^{-2\xi J_z}\Lambda e^{2\xi J_z} \quad (6)$$

and as such Λ is η-pseudo-Hermitian with $\eta = \exp(-2\xi J_z)$ (recalling that $\Lambda^\dagger = \eta\Lambda\eta^{-1}$). Furthermore η can be written as $\eta = OO^\dagger$ where $O^\dagger = O = \exp(-\xi J_z)$. Also, it can be easily shown that the operator Λ has a complete set of discrete bi-orthonormal eigenvectors $\{|\psi_m\rangle, |\phi_m\rangle\}$ where $|\psi_m\rangle$ are the eigenvectors of Λ with eigenvalue m and $|\phi_m\rangle$ are the eigenvectors of $\Lambda^\dagger$ with eigenvalue m. Thus the spectrum of Λ is completely real as it satisfies the relevant necessary and sufficient condition.

That Λ has real eigenvalues and these are given by $m = -j$ *to* $+j$ integrally spaced values may be seen from the following constructive procedure [8]. Consider the Wigner state $|j,m\rangle$ which is an eigenstate of J^2 and J_z belonging to eigenvalues $j(j+1)$ and m respectively. Rotate this state by the angle $\pi/2$ about the Y-axis. This vector, viz. $\exp(-iJ_y\pi/2)|j,m\rangle$ will be an eigenstate of J_x belonging to the eigenvalue m. Construct a state $|\psi_m\rangle$ by operating on $\exp(-iJ_y\pi/2)|j,m\rangle$ by $\exp(\xi J_z)$ and inserting a normalization constant (N_m) as $\exp(\xi J_z)$ not being unitary does not preserve the norm. It is clear that $|\psi_m\rangle$ will be an eigenstate of Λ belonging to the eigenvalue m, as

$$\Lambda|\psi_m\rangle = e^{\xi J_z}J_x e^{-\xi J_z}N_m e^{\xi J_z}e^{-iJ_y\pi/2}|j,m\rangle = e^{\xi J_z}N_m m e^{-iJ_y\pi/2}|j,m\rangle = m|\psi_m\rangle. \quad (7)$$

Recalling the definition of the reduced Wigner d-matrix, corresponding to a rotation about the y-axis by an angle β, we have

$$R_y(\beta)|j,m\rangle = \sum_{m'} d^j_{m'm}(\beta)|j,m'\rangle \quad (8a)$$

where

$$d^j_{m'm}(\beta) = \langle j,m'|R_y(\beta)|j,m\rangle = \langle j,m'|e^{-i\beta J_y}|j,m\rangle = (-1)^{m'-m}\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!} \times \sum_k \frac{(-1)^k(\cos\frac{\beta}{2})^{2j-2k-m'+m}(\sin\frac{\beta}{2})^{2k+m'-m}}{k!(j-m'-k)!(j+m-k)!(m'-m+k)!}, \quad (8b)$$

we recognize that

$$|\psi_m\rangle = N_m e^{\xi J_z}e^{-iJ_y\pi/2}|j,m\rangle = N_m e^{\xi J_z}\sum_{m'} d^j_{m'm}(\pi/2)|j,m'\rangle = N_m \sum_{m'=-j}^{+j} e^{\xi m'}d^j_{m'm}(\pi/2)|j,m'\rangle \quad (8c)$$

with the normalization constant given by

$$N_m^{-2} = \sum_{m'=-j}^{+j} e^{2\xi m'}d^j_{m'm}(\pi/2)d^j_{m'm}(\pi/2). \quad (8d)$$

At first sight this looks rather irksome involving a double sum as $d^j_{m'm}$ themselves are series [Eq. (8b)]. However, as shown in the Appendix employing the Addition Theorem and symmetry properties of the Wigner rotation matrices [9] in this and subsequent steps all such double sums may be avoided, and indeed to begin with the sum in equation (8d) leads to

$$N_m^{-2} = d^j_{mm}(2i\xi) \equiv \Delta \quad (8e)$$

where Δ can be obtained from equation (8b) and is the analytic continuation of the reduced Wigner d-matrix element for imaginary angles and is given by

$$\Delta = (j+m)!(j-m)! \times \sum_k \frac{(\cosh\xi)^{2j}(\tanh\xi)^{2k}}{(k!)^2(j-m-k)!(j+m-k)!}. \quad (9)$$

To investigate the properties of the squeezed state $|\psi_m\rangle$ using the techniques given above we need to find the relevant averages, standard deviations and correlations involving components of the angular momentum operator which we now proceed to carry out. From the eigenvalue equation itself [Eq. (7)], rewritten as

$$\Lambda|\psi_m\rangle = (J_x\cosh\xi + iJ_y\sinh\xi)|\psi_m\rangle = m|\psi_m\rangle \quad (10)$$

taking the scalar product with $\langle\psi_m|$ and equating real and imaginary parts one obtains right away the results

$$\langle\psi_m|J_x|\psi_m\rangle = m/\cosh\xi \quad (11a)$$

and

$$\langle\psi_m|J_y|\psi_m\rangle = 0. \quad (11b)$$

Taking the scalar product of equation (10) with its dual equation viz. $\langle\psi_m|\Lambda^\dagger\Lambda|\psi_m\rangle = m^2$ one obtains

$$\cosh^2\xi\langle\psi_m|J_x^2|\psi_m\rangle + \sinh^2\xi\langle\psi_m|J_y^2|\psi_m\rangle - \cosh\xi\sinh\xi\langle\psi_m|J_z|\psi_m\rangle = m^2. \quad (12a)$$

Operating with Λ in equation (10) and considering the resulting equation $\langle\psi_m|\Lambda^2|\psi_m\rangle = m^2$ and equating real and imaginary parts we have

$$\cosh^2\xi\langle\psi_m|J_x^2|\psi_m\rangle - \sinh^2\xi\langle\psi_m|J_y^2|\psi_m\rangle = m^2 \quad (12b)$$

and

$$\langle\psi_m|J_xJ_y + J_yJ_x|\psi_m\rangle = 0. \tag{12c}$$

The remaining correlators that we shall need are obtained by operating with J_z in equation (10) and taking the scalar product with $\langle\psi_m|$ to get

$$\langle\psi_m|J_z(J_x\cosh\xi + iJ_y\sinh\xi)|\psi_m\rangle = m\langle\psi_m|J_z|\psi_m\rangle$$

taking complex conjugate of which yields

$$\langle\psi_m|(J_x\cosh\xi - iJ_y\sinh\xi)J_z|\psi_m\rangle = m\langle\psi_m|J_z|\psi_m\rangle$$

whence addition of the two yields

$$\begin{aligned}\langle\psi_m|J_xJ_z + J_zJ_x|\psi_m\rangle &= \frac{2m}{\cosh\xi}\langle\psi_m|J_z|\psi_m\rangle \\ &\quad - \langle\psi_m|J_x|\psi_m\rangle\tanh\xi \\ &= \frac{2m}{\cosh\xi}\langle\psi_m|J_z|\psi_m\rangle - \frac{m\sinh\xi}{\cosh^2\xi}\end{aligned} \tag{12d}$$

where in the last step we have used equation (11a) and subtraction of the two yields

$$\langle\psi_m|J_yJ_z + J_zJ_y|\psi_m\rangle = 0.$$

The matrix elements that remain to be evaluated are worked out in the Appendix and we have

$$\langle\psi_m|J_z|\psi_m\rangle = \frac{1}{2\Delta}\frac{d\Delta}{d\xi} \tag{12e}$$

and

$$\langle\psi_m|J_z^2|\psi_m\rangle = \frac{1}{4\Delta}\frac{d^2\Delta}{d\xi^2} \tag{12f}$$

where Δ is given by equation (9). Finally, equations (12a), (12b) and (12e) together give

$$\langle J_x^2\rangle = \frac{m^2}{\cosh^2\xi} + \frac{\tanh\xi}{4\Delta}\frac{d\Delta}{d\xi}. \tag{12g}$$

To find $d^2\Delta/d\xi^2$ it is most convenient (as shown in the Appendix) to use the differential equation satisfied by $D^j_{m'm}(\alpha,\beta,\gamma)$ familiar from the quantum mechanics of the symmetric top. This leads to

$$\begin{aligned}\frac{d^2\Delta}{d\xi^2} &= 4j(j+1)\Delta - 4\frac{m^2\Delta}{\cosh^2\xi} - 2\coth 2\xi\,\frac{d\Delta}{d\xi} \\ &= 4j(j+1)\Delta - 4\frac{m^2\Delta}{\cosh^2\xi} - \Gamma\frac{\cosh 2\xi}{\cosh^2\xi}\end{aligned} \tag{12h}$$

where in the last step, to avoid indetermine forms (as $\xi \to 0$), we have used

$$\frac{d\Delta}{d\xi} = \tanh\xi\,\Gamma \tag{12i}$$

with

$$\Gamma = 2j\Delta + 2\frac{\eta}{\cosh^2\xi} \tag{12j}$$

and

$$\eta = (\cosh\xi)^{2j}(j+m)!(j-m)! \times \sum_k \frac{(\tanh\xi)^{2k}}{(k+1)!k!(j-m-1-k)!(j+m-1-k)!}. \tag{12k}$$

We notice that $\langle J_xJ_z + J_zJ_x\rangle = 0$ for $\xi = 0$, thus, showing that all the three correlators vanish in the case of a thermal field acting on the spin system, as it should.

3 Spin squeezing

We are now in a position to study the squeezing characteristics of the system represented by the wave function $|\psi_m\rangle$. Before doing so, as explained by the equations (3), we have to rotate in the xy-plane about the z-axis by an angle ϕ to the $x'y'$-plane and, then, rotate the $x'z$-plane about the y'-axis by an angle θ. This brings the z'-axis along the mean angular momentum vector as represented by the equation (3a). In the present case, since $\langle J_y\rangle = 0$, the angles are given by

$$\tan\theta = \frac{\langle J_x\rangle}{\langle J_z\rangle}, \qquad \phi = 0$$

where we have used the equation (3a). Thus the spin components in the two frames are related by

$$J_{x'} = J_x\cos\theta - J_z\sin\theta \tag{13a}$$

$$J_{y'} = J_y \tag{13b}$$

and

$$J_{z'} = J_x\sin\theta + J_z\cos\theta \tag{13c}$$

along with the mean spin vector given by equation (3a). It is easy to get the variance in the y' component from equations (12a) and (12b) which is

$$(\Delta J_{y'})^2 = \left\langle {J_{y'}}^2\right\rangle - \left\langle J_{y'}\right\rangle^2 = \left\langle {J_{y'}}^2\right\rangle = \frac{\langle J_z\rangle}{2\tanh\xi}. \tag{14}$$

We see that there is no correlation involved in the variance $(\Delta J_y')^2$ and, indeed, there is no squeezing in the y'-component which we show below. On the other hand, the variance in the x'-component given by

$$\begin{aligned}(\Delta J_{x'})^2 &= \langle J_x^2\rangle\cos^2\theta + \langle J_z^2\rangle\sin^2\theta \\ &\quad - \langle J_xJ_z + J_zJ_x\rangle\sin\theta\cos\theta\end{aligned} \tag{15a}$$

does involve quantum mechanical correlations. The averages appearing on the right-hand side of the above equations have been evaluated in equations (12a–12g). Substituting them, we get

$$\begin{aligned}(\Delta J_{x'})^2 =& \left[\frac{m^2}{\cosh^2\xi} + \frac{\tanh^2\xi}{4}\left(\frac{\Gamma}{\Delta}\right)^2\right]^{-1}\left[\left(\frac{\tanh^2\xi}{4}\right)^2\right. \\ &\left.\times\left(\frac{\Gamma}{\Delta}\right)^3 + \frac{j(j+1)m^2}{\cosh^2\xi} - \frac{m^2}{4\cosh^4\xi}\frac{\Gamma}{\Delta}\right] \\ &- \frac{m^2}{\cosh^2\xi}.\end{aligned} \tag{15b}$$

We notice immediately that $(\Delta J_{x'})^2$ is symmetric about $m = 0$ since Γ/Δ is symmetric [Eqs. (9, 12j, 12k)].

For $m = \pm j$, the variances are

$$(\Delta J_{x'})^2 = (\Delta J_{y'})^2 = \frac{j}{2}, \tag{16}$$

that is, they are independent of the squeeze parameter ξ and the state is in a minimum uncertainty state. This is identical to what happens with the Wigner states $|j, \pm j\rangle$. However, for $|m| < j$, we find the state $|\psi_m\rangle$ is spin squeezed. The degree of squeezing is a function of ξ and it vanishes as $\xi \to 0$. This is expected since $\xi = 0$ represents ordinary thermal field and it is well-known that thermal fields cannot bring about squeezing. In such a situation, the variances $(\Delta J_{x'})^2$ and $(\Delta J_{y'})^2$ in equations (14) and (15a) reduce to

$$(\Delta J_{x'})^2 = (\Delta J_{y'})^2 = \frac{1}{2}\left[j(j+1) - m^2\right] \tag{17}$$

as in the case of Wigner states $|j, m\rangle$ and again reduces to a minimum uncertainty state at $m = \pm j$. Thus, the interesting regime of spin squeezing is $|m| < j$ with $\xi \neq 0$.

As per equation (3c), we define spin squeezing parameter in the y' component [10]

$$Q = \sqrt{\frac{2}{|\langle \mathbf{J} \rangle|}}\, \Delta J_{y'} \tag{18a}$$

and, hence, the condition for squeezing is $Q < 1$. Since, $\langle J_y \rangle = 0$, we have

$$Q = \frac{1}{\sqrt{\tanh \xi}}\left[\frac{\langle J_z \rangle}{\sqrt{\langle J_x \rangle^2 + \langle J_z \rangle^2}}\right]^{1/2} \tag{18b}$$

where we have used equation (14). For $m = 0$, we have $\langle J_x \rangle = 0$ and, so, $Q \geq 1$ showing that there is no squeezing. Equation (18b) can be cast in the form

$$Q = \left[\frac{1}{\sqrt{\frac{4m^2\Delta^2}{\Gamma^2 \cosh^2 \xi} + \tanh^2 \xi}}\right]^{1/2}. \tag{18c}$$

It is now easy to see that, for $m \neq 0$, the denominator in the expression for Q satisfies

$$\frac{4m^2\Delta^2}{\Gamma^2 \cosh^2 \xi} + \tanh^2 \xi \leq 1.$$

Thus, we find that the noise in the y'-component cannot be quenched. However, the x'-quadrature is squeezed as we see below. We plot the spin squeezing parameter [10]

$$S = \sqrt{\frac{2}{|\langle \mathbf{J} \rangle|}}\, \Delta J_{x'} \tag{19}$$

as a function of ξ in Figure 1. According to the definition in equation (3c), $S < 1$ indicates squeezing. We notice in Figure 1 that S starts from the value with $\Delta J_{x'}$ for $\xi = 0$, as given in equation (17), and reaches a minimum after crossing the line $S = 1$ as ξ increases. But, further increase of ξ beyond this point makes S move upwards and we notice $S \to 1$ as $\xi \to \infty$.

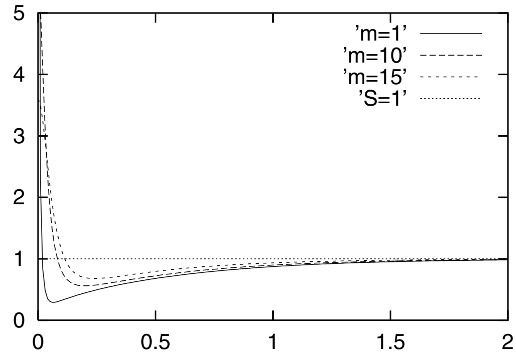

Fig. 1. Variation of S as a function of the radiation field squeeze parameter ξ. Note that $S > 1$ for $\xi = 0$. $j = 20$.

It may be noted here that, as mentioned earlier, the rotation from the unprimed to the primed axes does not bring any new correlations. Indeed, we notice

$$\langle J_{x'} J_{y'} + J_{y'} J_{x'} \rangle = 0,$$
$$\langle J_{y'} J_{z'} + J_{z'} J_{y'} \rangle = 0,$$

and

$$\langle J_{x'} J_{z'} + J_{z'} J_{x'} \rangle = \frac{m}{\cosh \xi} \tanh \xi \left[\frac{\Gamma}{\Delta} + \left\{\frac{m^2}{\cosh^2 \xi} + \frac{\tanh^2 \xi}{4}\left(\frac{\Gamma}{\Delta}\right)^2\right\}^{-1} \left\{\frac{(2\cosh^2 \xi - 1)}{4\cosh^2 \xi} \times \left(\frac{\Gamma}{\Delta}\right)^2 - j(j+1)\frac{\Gamma}{\Delta} + \frac{m^2}{\cosh^2 \xi}\right\}\right].$$

As expected, the above correlation goes to zero as $\xi \to 0$.

It would not be out of place if we investigate whether the uncertainty product

$$U = \frac{2}{|\langle \mathbf{J} \rangle|}\, \Delta J_{x'} \Delta J_{y'} \tag{20}$$

attains its minimum value in the range of ξ for which $S < 1$. We notice that $U \to 1$ as $|\langle \mathbf{J} \rangle| \to j$. The state $|\psi_m\rangle$ has

$$|\langle \mathbf{J} \rangle| = \langle J_{z'} \rangle = \left[\frac{m^2}{\cosh^2 \xi} + \frac{\tanh^2 \xi}{4}\left(\frac{\Gamma}{\Delta}\right)^2\right]^{1/2} \tag{21}$$

where Δ and Γ are given by equations (9) and (12j) respectively. We notice that $|\langle \mathbf{J} \rangle| = m$ for $\xi = 0$. As ξ is increased, the first term under the square root in equation (21) approaches zero and, from equations (9), (12j), and (12k), we find that $\Gamma/\Delta \to 2j$ as $\xi \to \infty$. Thus we see that $|\langle \mathbf{J} \rangle| = j$ for large ξ. These characteristics can be easily noticed in Figure 2. We further notice in Figures 1 and 2 that $|\langle \mathbf{J} \rangle| \cong j$ about the value of ξ at which the

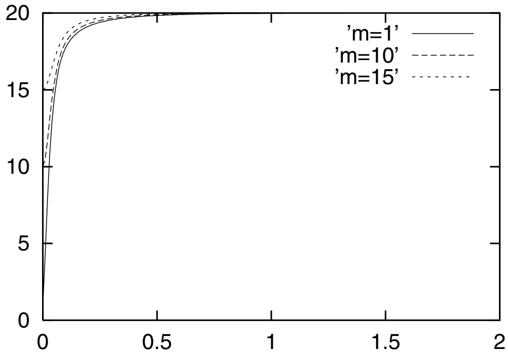

Fig. 2. Change in mean spin vector $|\langle \mathbf{J} \rangle|$ as a function of ξ. $j = 20$.

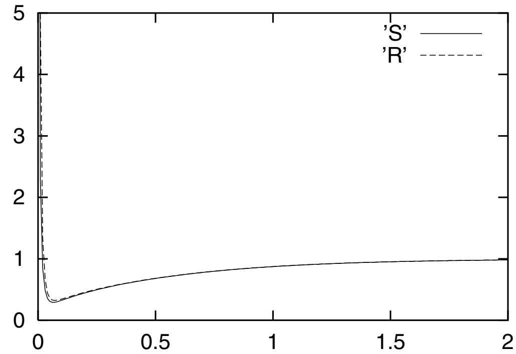

Fig. 3. Comparison of spin squeezing S with spectroscopic squeezing R. Both S and R are plotted as a function of ξ. $j = 20$ and $m = 1$.

curve for S goes below 1. It may have significance in the definition of spectroscopic squeezing which we discuss in the next section.

Figure 1 further indicates that the degree of squeezing increases with decreasing m. However, the squeezing depends on the difference $j - |m|$. Bigger this difference, better is the squeezing. To understand why this is so, we observe in equation (8c) that $|\psi_m\rangle$ has been generated from the Wigner state $|j, m\rangle$. The state has correlations among individual dipoles (spins) [11] which is proportional to $j^2 - m^2$ which is maximum at $m = 0$. It decreases as m increases and reaches zero at $m = \pm j$. These correlations obviously reflect in the wave function of the atomic (spin) system $|\psi_m\rangle$ and, thus, are at the root of spin squeezing. This also explains the absence of squeezing for $m = \pm j$ [Eq. (16)].

4 Conclusion

We have discussed the transfer of squeezing from the radiation field in a squeezed vacuum state to a system of N two-level atoms interacting with it. The interaction puts the atoms in a state which are eigenstates of the pseudo-Hermitian operator Λ [Eq. (6)] with real eigenvalues m. We have shown that, for large ξ, the state $|\psi_m\rangle$ satisfies $U = 1$ [Eq. (20)] with its x-quadrature spin squeezed. How large ξ should be depends on m since the mean spin vector $|\langle \mathbf{J} \rangle|$ is a function of ξ and m [Eq. (21)].

The present study shows the importance of pseudo-Hermitian operators in the present practical context. This is due to the fact that quantum state transfer from squeezed light to spin system is now experimental observable [12]. Spin squeezing is a subject of vigorous studies [12–15] nowadays. Among these studies, Wineland and others [13] preferred to present this squeezing slightly differently and called it the spectroscopic squeezing and is related to spin squeezing by

$$R = \sqrt{\frac{j}{|\langle \mathbf{J} \rangle|}}\, S. \tag{22}$$

We see that $R = S$ whenever the mean spin length $|\langle \mathbf{J} \rangle| = j$. Otherwise, the system may not show spectroscopic squeezing even if $S < 1$. We give a comparison in Figure 3 where we notice that the difference in R and S are quite small in the present system.

It is a matter of choice between R and S depending on which one is appropriate in a given experimental investigation. For example, in the atomic beam experiment by Wineland and others [13], it was R which was of importance there; whereas Kuzmich et al. [15] required spin squeezing in their quantum nondemolition measurement. The important fact to note here is that the eigenstates of the pseudo-Hermitian operator Λ can be traced by using the condition either $S < 1$ or $R < 1$.

Appendix

As is well-known the Wigner D^j matrices are the irreducible representations of the rotation group and encode the manner in which an angular momentum state $|j, m\rangle$ transform under a rotation $R(\alpha, \beta, \gamma)$ by the Eulerian angles, viz.,

$$R(\alpha, \beta, \gamma)|j, m\rangle = \sum_{m'} D^j_{m'm}(\alpha, \beta, \gamma)|j, m'\rangle. \tag{A.1}$$

Rotation about the y-axis by an angle β is given by the reduced Wigner matrix [see Eq. (8a)] and

$$d^j_{m'm}(\beta) = D^j_{m'm}(\alpha = 0, \beta, \gamma = 0). \tag{A.2}$$

For the evaluation of the summation contained in equation (8d) we shall need the addition formula [9] for $d^j_{mm'}$ (which is a generalization of the more well-known spherical harmonic addition formula):

$$\sum_{m'} d^j_{mm'}(\beta_1) d^j_{m'm''}(\beta_2) e^{-im'\phi} = e^{-im\alpha} d^j_{mm''}(\beta) e^{-im''\gamma} \tag{A.3}$$

with α, β and γ given by

$$\cot\alpha = \cos\beta_1 \cot\phi + \cot\beta_2 \frac{\sin\beta_1}{\sin\phi}, \tag{A.3a}$$

$$\cos\beta = \cos\beta_1 \cos\beta_2 - \sin\beta_1 \sin\beta_2 \cos\phi, \tag{A.3b}$$

and

$$\cot\gamma = \cos\beta_2 \cot\phi + \cot\beta_1 \frac{\sin\beta_2}{\sin\phi}. \tag{A.3c}$$

To make the summation occurring in equation (8d) amenable to this application of the theorem we rewrite it thus:

$$\begin{aligned} N_m^{-2} &= \sum_{m'} e^{2\xi m'} d^j_{m'm}(\pi/2) d^j_{m'm}(\pi/2) \\ &= \sum_{m'} e^{2\xi m'} d^j_{mm'}(-\pi/2) d^j_{m'm}(\pi/2) \end{aligned} \tag{A.4}$$

where we have used the symmetry of the reduced Wigner matrix $d^j_{mm'}(-\beta) = d^j_{m'm}(\beta)$. Comparing equation (A.4) with equation (A.3) we have $\phi = 2i\xi$, $\beta_1 = -\pi/2$, $\beta_2 = +\pi/2$ and $m'' = m$ implying via equations (A.3a–A.3c) that $|\alpha| = |\gamma| = \pi/2$ and accordingly

$$N_m^{-2} = d^j_{mm}(2i\xi)$$

which is equation (8e) of the text.

To handle the second derivative of the Wigner reduced matrix occurring in the calculation of the expectation value of J_z^2 in the squeezed state [see Eq. (12f)] we go back to the differential equation satisfied by $D^j_{mm'}(\alpha,\beta,\gamma)$ familiar from the quantum mechanics of an anisotropic rotor

$$\left[-\frac{1}{\sin\beta}\frac{\partial}{\partial\beta}\sin\beta\frac{\partial}{\partial\beta} + \frac{m^2 - 2mm'\cos\beta + m'^2}{\sin^2\beta}\right] \times D^j_{mm'}(\alpha,\beta,\gamma) = j(j+1) D^j_{mm'}(\alpha,\beta,\gamma).$$

Putting $m' = m$ and $\alpha = \gamma = 0$ we obtain

$$\frac{d^2}{d\beta^2} d^j_{mm}(\beta) = -j(j+1) d^j_{mm}(\beta) + m^2 \sec^2(\beta/2)\, d^j_{mm}(\beta) - \cot\beta \frac{d}{d\beta} d^j_{mm}(\beta)$$

wherein putting $\beta = 2i\xi$ and using the definition of Δ [Eq. (8e)] we obtain equation (12h).

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